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High-precision epsilon expansions of single-mass-scale four-loop vacuum bubbles

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Abstract: In this article we present a high-precision evaluation of the expansions in $\epsilon = (4-d)/2$ of (up to) four-loop scalar vacuum master integrals, using the method of difference equations developed by S. Laporta. We cover the complete set of 'QED-type' master integrals, i.e. those with a single mass scale only (i.e. $m_i \in \{0, m\}$) and an even number of massive lines at each vertex. Furthermore, we collect all that is known analytically about four-loop 'QED-type' masters, as well as about *all* single-mass-scale vacuum integrals at one-, two- and three-loop order.

Keywords: Standard Model, NLO Computations, QCD.

Contents

1. Introduction

Higher-order perturbative computations have become a necessity in many areas of theoretical physics, be it for high-precision tests of QED, QCD and the standard model, or for studying critical phenomena in condensed matter systems.

Most recent investigations employ a highly automated approach, utilizing algorithms that can be implemented on computer algebra systems, in order to handle the growing numbers of diagrams as well as integrals which occur at higher loop orders.

Computations can be divided into four key steps. First, the complete set of diagrams including symmetry factors has to be generated. For a detailed description of an algorithm for this step for the case of vacuum topologies, see ref. [1]. Second, after specifying the Feynman rules, the color- and Lorentz-algebra has to be worked out. Third, within dimensional regularization, massive use of the integration-by-parts (IBP) technique [2] to derive linear relations between different Feynman integrals in conjunction with an ordering prescription [3] can be used to reduce the (typically large number of) integrals to a basis of (typically a few) master integrals. Practical notes as well as a classification of vacuum master integrals are given in ref. [4]. Fourth, the master integrals have to be solved, either fully analytically, or in an expansion around the space-time dimension d of interest. It is the fourth step that we wish to address here.

A very important subset of master integrals are fully massive vacuum (bubble) integrals, since they constitute a main building block in asymptotic expansions (see e.g. ref. [5]). They are also useful for massless theories, when a propagator mass is introduced as an intermediate infrared regulator [6]. In four dimensions, this class of master integrals has been given up to the 4-loop level in ref. [7]. As an application, these integrals are vital for computing the 4-loop QCD beta-function and anomalous dimensions [8]. In lower dimensions, perturbative results are needed for applications in condensed matter systems, as well as in the framework of dimensionally reduced effective field theories for thermal QCD, where recent efforts have made four-loop contributions an issue [9]. We have recently extended the work of ref. [7], to give the complete set of fully massive vacuum master integrals in three dimensions, again up to the 4-loop level [10].

The next larger set of scalar vacuum master integrals are those in which there is only one mass-scale m, i.e. the propagators $1/(p_i^2 + m_i^2)$ have masses $m_i \in \{0, m\}$. These integrals are needed for problems with widely separated mass scales, in which one then sets the masses of all heavy particles to m and those of all light particles to zero. As a well-defined subset of these single-mass-scale integrals, we here treat 'QED-type' vacuum integrals, i.e. those with an even number of massive lines at each vertex, at the 4-loop level. A recent application is in the computation of heavy-quark vacuum polarization [11].

The complete set of 'QED-type' vacuum master integrals up to the 4-loop level has already been identified in ref. [4]. The main purpose of this work is to numerically compute this set in terms of a high-precision ϵ-expansion in $d = 4 - 2\epsilon$ dimensions, and to present new analytic results for some low-order (in ϵ) coefficients. Furthermore, we have made an attempt to collect all presently known analytic results on 4d single-mass-scale vacuum integrals, up to four loops, in a coherent notation.

The plan of the paper is as follows. In section 2, we give a brief review of the method of difference equations applied to vacuum integrals. In section 3, we discuss the actual implementation of the algorithm. In section 4, we display our numerical results for the truncated power series expansions in ϵ of our master integrals, up to the four-loop level, in $d = 4 - 2\epsilon$. In section 5, we discuss one case of a master integral which we needed to solve via a Laplace transform of its difference equation. In section 6, we list analytic results [12].

2. The evaluation of master integrals through difference equations

The method we have chosen to compute the coefficients of the truncated power series expansions of the master integrals is based on constructing difference equations for the integrals and then solving them numerically using factorial series. This approach has recently been developed in ref. [3], and below we briefly summarize its basic concepts following the notation of the original paper, which contains a much more detailed presentation of the subject. While the method is completely general as it applies to arbitrary kinematics, masses and topologies [13], our brief summary is somewhat adapted to the specific case of vacuum integrals.

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The main idea is to attach an arbitrary power x to one of the massive[1] lines of a master integral U,

$$U(x) \equiv \int \frac{1}{D_1^x D_2 \cdots D_N}\,, \tag{2.1}$$

where the $D_i = (p_i^2+m_i^2)$ denote inverse scalar propagators. In our case the mass parameter m_i has only two values, 0 and m, the latter of which we set to 1, noting that it can be restored in the end as a trivial dimensional pre-factor of each integral. The original master integral is then just $U = U(1)$. Depending on the symmetry properties of the integral, there can be different choices for the 'special' line with the arbitrary power x, but in the limit $x = 1$ they all reduce to the original master integral U. This degeneracy can (and will later) be used for non-trivial checks of the method.

Employing IBP identities in a systematic way, it is possible to derive a linear difference equation obeyed by the generalized master integral $U(x)$,

$$\sum_{j=0}^{R} p_j(x) U(x+j) \,=\, F(x)\,, \tag{2.2}$$

where R is a finite positive integer and the coefficients p_j are polynomials in x (and the space-time dimension d). The function F on the r.h.s. is a linear combination of functions analogous to $U(x)$ but derived from 'simpler' master integrals, i.e. integrals containing a smaller number of loops and/or propagators.

The general solution of this kind of an equation is the sum of a special solution to the full equation, $U_0(x)$, and the solutions to the homogeneous equation ($F = 0$),

$$U(x) \,=\, U_0(x) + \sum_{j=1}^{R} U_j(x)\,, \tag{2.3}$$

where each ($j = 0, \ldots, R$)

$$U_j(x) = \mu_j^x \sum_{s=0}^{\infty} a_j(s) \frac{\Gamma(x+1)}{\Gamma(x+1+s-K_j)} \tag{2.4}$$

is a factorial series.[2] Substituting this form into eq. (2.2), one obtains the coefficients μ_j and K_j (the latter being a function of d), as well as recursion relations for the x-independent coefficients $a_j(s)$ (being functions of d as well) for each solution. For the homogeneous solutions, these recursion relations relate all coefficients with $s > 0$ to their (in principle arbitrary) value at $s = 0$, $a_j(s) = c_j(s)\, a_j(0)$, where the $c_j(s)$ are rational functions of d. For the special solution, all $a_0(s)$ are on the other hand completely fixed in terms of the inhomogeneous part $F(x)$, consisting of 'simpler' integrals which are assumed to be already known in terms of their factorial series expansions.

[1]The massiveness is crucial in order to avoid problems with the infrared behavior of the integral.

[2]For a rigorous definition of the concept as well as a motivation for this kind of an Ansatz, we refer the reader to ref. [3].

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What clearly remains to be done is to fix the x- and s-independent constants $a_j(0)$, $j \neq 0$, in order to determine the weights of the different homogeneous solutions. To this end, it is most useful to study the behavior of $U(x)$ at large x. Writing the integral in the form

$$U(x) = \int \frac{1}{(p_1^2+1)^x}\, g(p_1)\,, \tag{2.5}$$

it is easy to see that its large-x behavior is determined by the small-momentum expansion of the two-point function $g(p_1)$, which has one loop less than the original vacuum integral.

In the case of integrals for which the limit $g(0)$ is well-defined and non-zero, the calculation becomes particularly simple. Then the large-x limit of $U(x)$ factorizes into a one-loop bubble carrying the large power x and a lower-loop vacuum bubble $g(0)$, which corresponds to $U(x)$ with its 'special' line cut away,

$$\lim_{x\to\infty} U(x) = \left[\int \frac{1}{(p_1^2+1)^x}\right] \times \Big[g(0)\Big] \sim (1)^x x^{-d/2} g(0)\,. \tag{2.6}$$

A comparison with the large-x behavior of eqs. (2.3), (2.4), proportional to $\sum_j \mu_j^x a_j(0) x^{K_j}$, can now be used to fix the $a_j(0)$, of which maximally one will turn out to be non-zero for our set of integrals.

If on the other hand $g(0) = 0$, the treatment of the small-p_1 limit of this function becomes more involved. Fortunately, the massless lines of the sub-diagram — which were responsible for the vanishing of its value at zero external momentum in the first place — also make its analytic evaluation more straightforward. Performing a careful analysis of the subgraph, one always ends up with an integral of the type

$$\lim_{x\to\infty} U(x) \sim \int \frac{(p_1^2)^\alpha}{(p_1^2+1)^x}\,, \tag{2.7}$$

from which the calculation proceeds just as above providing us with the values of the $a_j(0)$.

Having the full solution at hand, we have in principle completed our task, as in the limit $x = 1$ we recover from $U(x)$ the value of the initial integral. Let us, however, add a couple of practical remarks here. What is still to be done is to perform the summation of the factorial series of eq. (2.4), which in practice means truncating the infinite sum at some large but finite $s_{\max}$. Studying the convergence behavior of these sums, one notices that even in the cases where they do converge down to $x \sim 1$, their convergence properties usually strongly decline with decreasing x. This means that in practical computations, where one aims at obtaining a maximal number of correct digits for $U(1)$ with as little CPU time as possible, the optimal strategy is to evaluate the integrals $U(x_{\max}+1), \ldots, U(x_{\max}+R)$ with the factorial series approach at some $x_{\max} \gg 1$ and then use the recurrence relation of eq. (2.2) to obtain the desired result at $x = 1$. The price to pay is, however, a loss of numerical accuracy at each 'pushdown' ($x \to x-1$) step due to possible cancellations, which makes the use of a very high $x_{\max}$ impossible. In practice the strategy is to determine an optimal value for the ratio $s_{\max}/x_{\max}$. To give an example, for the four-loop integrals of section 4 we have found that $s_{\max}/x_{\max} \sim 20\ldots40$ is a good value, while we used a range of $s_{\max} \sim 2000 \ldots 2500$. For a few special cases, for which additional numerical problems emerged, we were forced to limit the value of the parameters to roughly $s_{\max} \sim 200$ and $x_{\max} \sim 30$, which decreased the accuracy of the results significantly.

3. Implementation of the algorithm

As is apparent from the preceding section, there are three main steps involved in obtaining the desired numerical coefficients in the ϵ-expansion of each master integral: deriving the difference equations obeyed by each integral, solving them in terms of factorial series, and finally performing the ϵ-expansion and numerically evaluating the sum of eq. (2.4) (truncated at $s_{\max}$) to the precision needed. We will briefly address each of them in the following.

For the first step, we slightly generalized the IBP algorithm we had used for reducing generic 4-loop bubble integrals to master integrals, which follows the setup given in ref. [3], and whose implementation in FORM [14] is documented in ref. [4]. The main difference is an enlarged representation for the integrals, keeping track of the line which carries the extra powers x, as well as the fact that there are now two independent variables (d, x), requiring factorization (and inversion) of bivariate polynomials, as opposed to univariate polynomials in the original version.

Second, staying within FORM for convenience, we implemented routines that straightforwardly solve the difference equations in terms of factorial series, along the lines of ref. [3]. This is done starting with the simplest one-loop master integral, and working the way up to the most complicated (most lines) four-loop integral, ensuring that at each step, the 'simpler' terms constituting the inhomogeneous parts of the difference equation are already known. The output are then plain ascii files specifying each solution in the form of eq. (2.4) as well as containing recursion relations for the coefficients $a(s)$. Note that these first two steps are performed exactly, in d dimensions.

Third, once the recursion relations for the coefficients $a(s)$ were known, we used a Mathematica program to obtain their numerical values at each s to a predefined precision, and to perform the summation of the factorial series. While this procedure is in principle straightforward, there are some twists that we employed to help reduce the running times significantly, most of which are probably quite specific to our use of Mathematica. To avoid a rapid loss of significant digits in solving the recursion steps that relate each $a(s)$ to $a(0)$, especially those for the homogeneous coefficients, we first solved the relations analytically and only in the end substituted the numerical value (actually the truncated ϵ-expansion) of the first non-zero coefficient. In fact, we found Mathematica to operate quite efficiently with operations like multiplication of two truncated power series, so that we relied heavily on it. Furthermore, since — not surprisingly — the most time-consuming part in the summation of the series turned out to be the ϵ-expansion of Γ-functions, we achieved a notable speed-up by substituting the Γ-functions with large arguments by suitable products of linear factors times Γ-functions of smaller arguments. Finally, a vital step in avoiding an excessive loss in the depth of the ϵ-expansions when going from one integral to the next, was to apply the 'Chop' command to remove from the results and coefficients excess unphysical poles, whose coefficients were of the order of, say, 10^{-50} or less. In some cases we were in addition able to reduce the loss of precision in the pushdown steps by first analytically solving $U(1)$ as a function of $U(x_{\rm max})$, and only in the very end substituting the numerical value of the latter.

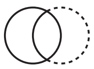
4. Numerical results

Below we list the Laurent expansions in $\epsilon = (4-d)/2$ of vacuum master integrals up to four loops. We use an intuitive graphical notation, in which each solid line represents a massive scalar propagator $1/(p^2+1)$ and a dashed line a massless one $1/p^2$. The integral measure we have chosen here is

$$\int_p \equiv \frac{1}{\Gamma(2+\epsilon)} \int \frac{\mathrm{d}^{4-2\epsilon}p}{\pi^{2-\epsilon}}\,, \tag{4.1}$$

which implies that the 1-loop tadpole is $J = \int_p \frac{1}{p^2+1} = \frac{-1}{\epsilon(1-\epsilon^2)} = -\sum_{n=0}^{\infty} \epsilon^{2n-1}$. In each case we provide the results to order ϵ^{10} keeping the accuracy at 50 significant digits for the 2- and 3-loop master integrals and at 40 for the 4-loop ones. There are two exceptions. For one of the 3-loop integrals (see eq. (4.6) below) the factorial series does not converge and hence the integral has to be treated by Laplace transform, see section 5. For one of the 4-loop integrals (see eq. (4.14) below) we only give the first seven ϵ-orders to 17 significant digits. To obtain more ϵ-orders and significant digits for all integrals listed here is merely a matter of additional CPU time.

We have produced numerical results for all single-mass-scale vacuum master integrals up to three loops (these are the master integrals entering the package of Avdeev [15] and MATAD [16]), and for all 'QED-type' vacuum master integrals at four loops. Here, we display only those numerical results which correspond neither to analytically solvable integrals (1 of 1 1-loop master, 1 of 2 2-loop masters, 3 of 12 3-loop masters, 2 of 10 4-loop masters, all of which are given analytically in section 6 below, and are listed in numerical form in the appendix), nor to fully massive cases (1 of 2 2-loop masters, 3 of 12 3-loop masters, which are given in section 9.3.1 of ref. [3]; 1 of 10 4-loop masters, which is given in eq. (4) of ref. [7]).



$$\begin{aligned}
= \; & + 1.0000000000000000000000000000000000000000000000000\ \epsilon^{-3} + \\
& + 0.75000000000000000000000000000000000000000000000000\ \epsilon^{-2} + \\
& + 2.8750000000000000000000000000000000000000000000000\ \epsilon^{-1} + \\
& + 1.8362912870512825038535151499626234054646567716807 - \\
& - 26.427828097688527267319254765120590367456377175480\ \epsilon - \\
& - 35.088051385481306364065961402117419432373775682177\ \epsilon^{2} - \\
& - 512.75537623727044689027104289971864365971796649684\ \epsilon^{3} - \\
& - 607.61494953927726782115473332930225595551912885034\ \epsilon^{4} - \\
& - 5868.5987295458313170081280447224279031237930577453\ \epsilon^{5} -
\end{aligned}$$

$$
\begin{aligned}
&- 6835.6108788455114123641492279253803965001408075543\ \epsilon^{6} - \\
&- 58194.090725773231428299235587057139067942816045554\ \epsilon^{7} - \\
&- 67435.335245041201792055506063164867635607825896649\ \epsilon^{8} - \\
&- 546094.78026628005592146280450252032502449454982782\ \epsilon^{9} - \\
&- 631563.41278231233491152773645513360004834876043263\ \epsilon^{10} + \\
&+ \mathcal{O}\left(\epsilon^{11}\right)
\end{aligned}
\tag{4.2}
$$

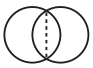
$$
\begin{aligned}
= &- 0.66666666666666666666666666666666666666666666666667\ \epsilon^{-3} - \\
&- 1.6666666666666666666666666666666666666666666666667\ \epsilon^{-2} - \\
&- 5.1290732088140381005700717333376095855758453867530\ \epsilon^{-1} - \\
&- 26.359970069205366659319388577532454678949629074714 - \\
&- 27.711175418518951962132692178901111387631205211816\ \epsilon - \\
&- 293.15661097603756640443665077615751632698451158842\ \epsilon^{2} - \\
&- 142.70296384808301760189570443963964069968530061393\ \epsilon^{3} - \\
&- 2882.1838924952422595902727649575335612132315437366\ \epsilon^{4} - \\
&- 801.64629651874722343778241421866459175486305997074\ \epsilon^{5} - \\
&- 26947.975116190322227046885628024191588708203470044\ \epsilon^{6} - \\
&- 5202.1954102253813787831194867097139379256134207497\ \epsilon^{7} - \\
&- 246612.58893683893330836716807041349553641918919392\ \epsilon^{8} - \\
&- 38662.312198716830334636275721442722552625311137805\ \epsilon^{9} - \\
&- 2235893.9169450155346378997842831790622571843918826\ \epsilon^{10} + \\
&+ \mathcal{O}\left(\epsilon^{11}\right)
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
= &+ 2.4041138063191885707994763230228999815299725846810\ \epsilon^{-1} - \\
&- 13.125546202841586242894146861604104971473328745577 + \\
&+ 58.026260003878655576719786597271170572487856789112\ \epsilon - \\
&- 215.15799420773251496754795359758001229751012774168\ \epsilon^{2} + \\
&+ 741.02167568175382570477503744319405056068752490840\ \epsilon^{3} - \\
&- 2422.8745603243623464433277838674972111388328822910\ \epsilon^{4} + \\
&+ 7691.0946660371679072096695375419004253731139722117\ \epsilon^{5} - \\
&- 23935.477541938694107878632636617038283240279946231\ \epsilon^{6} + \\
&+ 73567.948130076321368433008921329180208448045889323\ \epsilon^{7} - \\
&- 224259.22429731742234354745849077250951082319381698\ \epsilon^{8} + \\
&+ 679949.06185664528482517935972319009053625673803897\ \epsilon^{9} - \\
&- 2054250.6137900838709156880585181458050495843111626\ \epsilon^{10} + \\
&+ \mathcal{O}\left(\epsilon^{11}\right)
\end{aligned}
\tag{4.4}
$$

$$= +2.4041138063191885707994763230228999815299725846810\ \epsilon^{-1} -$$
$$-10.073203643096893062671213536841941862151359216063 +$$
$$+46.082030897278984204342981632818973100797752268016\ \epsilon -$$
$$-162.84321571472549604685427998929495247564452607777\ \epsilon^{2} +$$
$$+563.02541599052549921690912303391142482056503193963\ \epsilon^{3} -$$
$$-1822.8278416039379661792322993085062379900421439244\ \epsilon^{4} +$$
$$+5785.9815122286472118701238800303861843636492370028\ \epsilon^{5} -$$
$$-17968.847688521304415691142884872355494614421789034\ \epsilon^{6} +$$
$$+55216.376506037509642111329809657803343085975716148\ \epsilon^{7} -$$
$$-168240.56307714987438328576061703042187348961355271\ \epsilon^{8} +$$
$$+510052.27830760883492002666904963035268124951366379\ \epsilon^{9} -$$
$$-1540802.7858406456522499592944279183737583148997338\ \epsilon^{10} +$$
$$+\mathcal{O}\left(\epsilon^{11}\right) \tag{4.5}$$

$$= +2.40411380631919\ \epsilon^{-1} - 6.09209302191832 +$$
$$+35.8130598712514\ \epsilon - 104.744695525740\ \epsilon^{2} +$$
$$+394.7643404810\ \epsilon^{3} - 1200.978166746\ \epsilon^{4} + \mathcal{O}\left(\epsilon^{5}\right) \tag{4.6}$$

$$= +2.4041138063191885707994763230228999815299725846810\ \epsilon^{-1} -$$
$$-10.239350912945217732184803670827657230740659540460 +$$
$$+46.310233388509835938575195677581891346572247104081\ \epsilon -$$
$$-163.71903666846274587940160817767510251267798801563\ \epsilon^{2} +$$
$$+564.30910069499791449917891053192483169414448523830\ \epsilon^{3} -$$
$$-1825.8206691490586101339592917414250683553208417056\ \epsilon^{4} +$$
$$+5790.4830503256226500389801844087331481909799175043\ \epsilon^{5} -$$
$$-17977.329926014373828927297954140399343188401436964\ \epsilon^{6} +$$
$$+55229.228709840982549894271961274899889724333654202\ \epsilon^{7} -$$
$$-168262.33984881039469900336415583383277506556682677\ \epsilon^{8} +$$
$$+510085.27212468040781599787353617454516173782111015\ \epsilon^{9} -$$
$$-1540855.6379207615212777547545938657889645397412471\ \epsilon^{10} +$$
$$+\mathcal{O}\left(\epsilon^{11}\right) \tag{4.7}$$



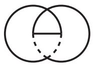
$$= -1.000000000000000000000000000000000000000\ \epsilon^{-4} -$$
$$-0.500000000000000000000000000000000000000\ \epsilon^{-3} -$$
$$-3.527777777777777777777777777777777777778\ \epsilon^{-2} -$$
$$-1.995370370370370370370370370370370370370\ \epsilon^{-1} -$$
$$-36.82021604938271604938271604938271604938 -$$
$$-19.87920801451107035069575635156380101575\ \epsilon -$$

$$
\begin{aligned}
&- 1809.001126638637160933894507798781706682\ \epsilon^{2} - \\
&- 941.2486498215135407529753614624254521594\ \epsilon^{3} - \\
&- 49114.80404240275263940837370626747663512\ \epsilon^{4} - \\
&- 25712.87944658606239888301931387195377680\ \epsilon^{5} - \\
&- 1014742.540337323108931396794699794706304\ \epsilon^{6} - \\
&- 533925.9315185165221824312193117157135164\ \epsilon^{7} - \\
&- 18513953.44519328478360685998151320048728\ \epsilon^{8} - \\
&- 9769845.146715270007449428486953016122496\ \epsilon^{9} - \\
&- 317669932.9515691976277658362596784695115\ \epsilon^{10} + \mathcal{O}\left(\epsilon^{11}\right)
\end{aligned}
\tag{4.8}
$$

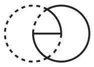
$$
\begin{aligned}
= &+ 0.2500000000000000000000000000000000000000\ \epsilon^{-4} + \\
&+ 0.5000000000000000000000000000000000000000\ \epsilon^{-3} + \\
&+ 1.000000000000000000000000000000000000000\ \epsilon^{-2} + \\
&+ 1.813369870537362855098298049824424939972\ \epsilon^{-1} - \\
&- 113.8224542836131461311762552843948945680 - \\
&- 33.70692008121875082746730709549318292582\ \epsilon - \\
&- 3800.131117795239883336446808648670131032 4\ \epsilon^{2} - \\
&- 724.2483980459868435529785916706580415218\ \epsilon^{3} - \\
&- 83243.75114211351557600351242603548310943\ \epsilon^{4} - \\
&- 9962.244874731471054690629554209449745080\ \epsilon^{5} - \\
&- 1556494.392681571176934758495668112116219\ \epsilon^{6} - \\
&- 125852.9269007094630774780949002157883896\ \epsilon^{7} - \\
&- 27026768.74324139691004925806420463865625\ \epsilon^{8} - \\
&- 1619900.985945231199760429618558131115494\ \epsilon^{9} - \\
&- 451968203.1707264233126870326577507342793\ \epsilon^{10} + \mathcal{O}\left(\epsilon^{11}\right)
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
= &+ 0.6666666666666666666666666666666666666667\ \epsilon^{-4} + \\
&+ 1.333333333333333333333333333333333333333\ \epsilon^{-3} + \\
&+ 3.333333333333333333333333333333333333333\ \epsilon^{-2} - \\
&- 2.922363183148830477868063138605600049253\ \epsilon^{-1} - \\
&- 52.50529739842769756973487794955803028226 - \\
&- 622.1548972708590376012515685880304077291\ \epsilon - \\
&- 1741.392944346262052260405956917114927201\ \epsilon^{2} - \\
&- 17196.12685330902582768340098554237636824\ \epsilon^{3} - \\
&- 35037.76438140725371856293904191777384497\ \epsilon^{4} - \\
&- 350040.6052285494016783912074340410365119\ \epsilon^{5} - \\
&- 619669.7756160060500505884016704452111642\ \epsilon^{6} -
\end{aligned}
$$

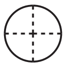
$$
\begin{aligned}
&- 6316632.0787940154696023419736840433315269\ \epsilon^{7} - \\
&- 10420684.666262760453830102149282840874 92\ \epsilon^{8} - \\
&- 107682720.99366560868072010024983134478 72\ \epsilon^{9} - \\
&- 171175743.73347858893160264977743595870 50\ \epsilon^{10} + \mathcal{O}\left(\epsilon^{11}\right) \qquad (4.10)
\end{aligned}
$$



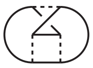
$$
\begin{aligned}
= &- 0.1666666666666666666666666666666666666667\ \epsilon^{-4} - \\
&- 0.8333333333333333333333333333333333333333\ \epsilon^{-3} - \\
&- 5.5353902364929276187330714948447833240 98\ \epsilon^{-2} - \\
&- 18.822113581793644430340846770470785193 65\ \epsilon^{-1} - \\
&- 25.331317096391036302979342976322191026 42 - \\
&- 692.62536813838592078022916113528113588 18\ \epsilon + \\
&+ 1304.4068271890231738355217316833894675 96\ \epsilon^{2} - \\
&- 17597.627617427671757963421102530404168 42\ \epsilon^{3} + \\
&+ 43608.684787250409737613215350228632506 02\ \epsilon^{4} - \\
&- 356925.79479522125852333853078041002649 07\ \epsilon^{5} + \\
&+ 939175.52089361334990007323081718815593 93\ \epsilon^{6} - \\
&- 6467516.5679311609823873248814609095954 34\ \epsilon^{7} + \\
&+ 17364082.003165434699469425440361345442 07\ \epsilon^{8} - \\
&- 110630064.05047189627992941089693644103 21\ \epsilon^{9} + \\
&+ 299555848.29671998418018456641121465444 29\ \epsilon^{10} + \mathcal{O}\left(\epsilon^{11}\right) \qquad (4.11)
\end{aligned}
$$



$$
\begin{aligned}
= &- 0.1666666666666666666666666666666666666667\ \epsilon^{-4} - \\
&- 0.8333333333333333333333333333333333333333\ \epsilon^{-3} - \\
&- 4.9343617849131304760332024140890583287 16\ \epsilon^{-2} - \\
&- 16.207285801770899358604236755797734280 42\ \epsilon^{-1} - \\
&- 66.260452670882537199501453227038221923 30 - \\
&- 375.81318076482585685908857397859878421 70\ \epsilon - \\
&- 558.86849802910566114977393270056485550 56\ \epsilon^{2} - \\
&- 8005.9092580523081313246072288360463816 31\ \epsilon^{3} - \\
&- 1919.4554723133574015275736202572694583 94\ \epsilon^{4} - \\
&- 151678.44738721749060371024342260853123 95\ \epsilon^{5} + \\
&+ 45367.137036765531189438846406449283123 37\ \epsilon^{6} - \\
&- 2666691.7714755542010537714438747777448 95\ \epsilon^{7} + \\
&+ 1469487.2593136417875424284807078317791 15\ \epsilon^{8} - \\
&- 44948193.387762776868968045732124024727 50\ \epsilon^{9} + \\
&+ 30458142.549833288759701864267625081695 87\ \epsilon^{10} + \mathcal{O}\left(\epsilon^{11}\right) \qquad (4.12)
\end{aligned}
$$



$$
\begin{aligned}
= &+ 5.184638775716849631656827432285170840285 \;\epsilon^{-1} - \\
&- 43.27615856932464061120186605318440014787 + \\
&+ 281.6878028207571441503294898731325032572 \;\epsilon - \\
&- 1513.439498357334510783856768928671867962 \;\epsilon^{2} + \\
&+ 7425.188218180801392788420314778806091908 \;\epsilon^{3} - \\
&- 34157.28328369996440389044714608216014037 \;\epsilon^{4} + \\
&+ 150927.1863992836861076852919478736657397 \;\epsilon^{5} - \\
&- 648212.2200729992344766408701706025837197 \;\epsilon^{6} + \\
&+ 2730180.324952706549742379412290749026850 \;\epsilon^{7} - \\
&- 11339683.06464037751511038174569497508966 \;\epsilon^{8} + \\
&+ 46630964.11789747669532801502683952077042 \;\epsilon^{9} - \\
&- 190369535.4103429881202484621694879170262 \;\epsilon^{10} + \mathcal{O}\left(\epsilon^{11}\right) \qquad (4.13)
\end{aligned}
$$



$$
\begin{aligned}
= &+ 1.80887954620833474 - 12.7814836099524403 \;\epsilon + \\
&+ 71.046049240835262 \;\epsilon^{2} - 334.59648933739741 \;\epsilon^{3} + \\
&+ 1467.5837602507405 \;\epsilon^{4} - 6165.5621168597119 \;\epsilon^{5} + \\
&+ 25329.619267580422 \;\epsilon^{6} + \mathcal{O}\left(\epsilon^{7}\right) \qquad (4.14)
\end{aligned}
$$

Just as in the three-dimensional case [10], we have performed various checks on our results. These can be divided into two categories: first, we had to make sure that the difference equation, eq. (2.2), as well as the various parts of its solution, eqs. (2.3), (2.4), were in principle correct, and second, that we have reached the desired accuracy in the numerical part of our computation. For the first task, it was in general enough to ensure that the first few ϵ-orders we obtained for each integral coincided with the existing analytic results. Here the main difference to our previous three-dimensional computation laid in the fact that while at $d = 3 - 2\epsilon$ most of the integrals were either finite or their expansions started with a $1/\epsilon$ term, we now encountered in many cases (often analytically calculable) divergent terms up to $1/\epsilon^4$ order. The analytic results relevant to our graphs that we have found in the literature, as well as a few new ones, are collected in section 6. We have found agreement in all cases.

The comparisons with existing analytic results also provide an easy and reliable method to inspect the accuracy of the numerical results, since the number of correct digits usually stays roughly constant when moving from one ϵ-order to the next. Just as in our previous work [10], other methods we have employed to assess the accuracy question include comparing the results obtained by raising topologically inequivalent lines in a single integral to a higher power and analyzing the convergence properties of the factorial series, i.e. checking the stability of our results with respect to varying $s_{\max}$. The results given in the preceding section have been observed to be stable to at least the number of digits shown.

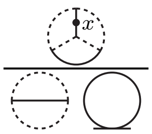
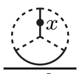
One might be concerned about the rapid growth with increasing ϵ-orders of most of the coefficients. This is, as was pointed out in ref. [7], caused by poles that the integrals (seen as functions of d) develop near $d = 4$, e.g. at $d = 7/2, 3$, etc. It is to be expected that factoring out the first few of these nearby poles in each case will improve the apparent convergence in ϵ considerably.

In principle, having a method at hand that is capable of generating coefficients to very high accuracy, even to a couple of hundred digits, one could now use the algorithm PSLQ [17] combined with an educated guess of the number content of some of the yet-unknown constant terms, in order to search for analytic representations of the numerical results. We have not made any systematic attempts in that direction, since the numerical accuracy of our results should be sufficient for all practical purposes. However, for a few leading coefficients we have successfully applied this method. The analytic values are given in section 6.

5. Laplace transform

As already mentioned in the above, we have encountered one case where the method of computing the ϵ-expansion via a factorial series representation does not work (or, more precisely, does not converge), namely for the 3-loop integral of eq. (4.6). Let us take this specific example as an opportunity to finally display a difference equation like eq. (2.2) in full detail and exhibit, following ref. [3], one method other than factorial series to solve it.

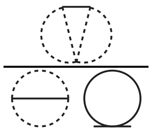
Defining the integral

$M_2(x) \equiv$  $=$  $\frac{1}{J^3}\,\frac{2^{d-2}\Gamma(\frac{1}{2})}{\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})}\,, \qquad (5.1)$

where the dot with the label x means that the corresponding propagator is raised to the x-th power, the difference equation eq. (2.2) it satisfies is of second order and reads

$$\begin{aligned} 0 = & -2(x+1)M_2(x+2) + 3\left(x+2-\frac{d}{2}\right)M_2(x+1) - (x+3-d)M_2(x) + \\ & + \frac{\Gamma(x+5-\frac{3d}{2})}{\Gamma(x+1)}\frac{3-d}{\Gamma(5-\frac{3d}{2})}M_2(0) + \frac{\Gamma(x+3-d)}{\Gamma(x+1)}\frac{1}{\Gamma(2-d)} - \\ & - \frac{\Gamma(x+2-\frac{d}{2})}{\Gamma(x+1)}\frac{2}{\Gamma(1-\frac{d}{2})} + \frac{\Gamma(x+5-\frac{3d}{2})\Gamma(x+3-d)}{\Gamma(x)\Gamma(x+7-2d)}\frac{2}{\Gamma(1-\frac{d}{2})}\,, \end{aligned} \tag{5.2}$$

with boundary conditions $M_2(x \gg 1) \sim x^{-\frac{d}{2}}$ (cf. eq. (2.6)) and

$M_2(0) =$  $= -\frac{\Gamma(\frac{3d}{2})\Gamma(1-\frac{3d}{2})\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2})}{\Gamma(d)\Gamma(d-2)\Gamma(1-d)}\,. \qquad (5.3)$

We would like to know the master integral $M_2(1)$, or at least its ϵ-expansion in $d = 4-2\epsilon$ dimensions. Note that only the first two terms of that expansion are known, cf. eq. (6.32) below. Formally, it is of course possible to solve eq. (5.2) in terms of factorial series, following the recipe sketched in section 2. However, it turns out that the series does not converge in this (and only this, of all cases treated in this paper) case, such that in practice a different method of solving the difference equation is needed.

One way to tackle eq. (5.2) could be the iterative method used e.g. in ref. [18] (cf. eq. (11)ff therein): expand in ϵ, make Ansätze for the ϵ-coefficients of M_2 in terms of (sums of multiple) harmonic sums with unknown constants, write all ϵ-expansions of the Gamma functions in terms of harmonic sums, then rewrite everything in terms of a unique basis, and finally fix the constants by comparing coefficients. Unfortunately, there does not seem to exist an algorithm yet that automatizes the choice of Ansatz, hence requiring a fair amount of hand-work. For the basic literature on harmonic sums, see the references of ref. [18].

Another way of tackling eq. (5.2) is to transform it to a differential equation, which should then be solved by analytical or numerical methods, or by a combination of both. This is what we will do in the following, and this is how we have obtained the numerical values given in eq. (4.6).

We can use the Ansatz $M_2(x) = \int_0^1 dt\, t^{x-1} v(t)$ to Laplace transform the difference equation for M_2 into a first order differential equation $\Phi_0(t)v(t) - t\Phi_1(t)v'(t) = w(t)$, where $\Phi_0(t) = 3-d-3(1-\frac{d}{2})t - 2t^2$ and $\Phi_1(t) = (1-t)(1-2t)$ [3].

The homogeneous equation is solved by $v_H(t) = c_H t^{3-d}(1-t)^{\frac{d}{2}-2}(1-2t)^{\frac{d}{2}-2}$, which however makes $M_2(x \gg 1)$ grow too fast at large x (it would grow like $x^{1-\frac{d}{2}}$, in conflict with the large-x boundary condition), such that $c_H \equiv 0$ and hence ${M_2}^H(x) = 0$.

For solving the inhomogeneous equation, note that the inhomogeneous piece has four terms $w(z) = \sum_{j=1}^4 w_j(z)$, which correspond to the last four terms of eq. (5.2), written as $T_j(x) = \int_0^1 dz\, z^{x-1} w_j(z)$. For $j = 1,2,3$ we therefore have $w_j(z) = \frac{b_j}{\Gamma(1-a_j)} z^{a_j}(1-z)^{-a_j}$, where $\vec{a} = (5-\frac{3d}{2}, 3-d, 2-\frac{d}{2})$ and $\vec{b} = (\frac{(3-d)M_2(0)}{\Gamma(5-\frac{3d}{2})}, \frac{1}{\Gamma(2-d)}, -\frac{2}{\Gamma(1-\frac{d}{2})})$. For w_4, we know that it satisfies

$$\int_0^1 dz\, z^{x-1} w_4(z) = T_4(x) = \frac{\Gamma(x+5-\frac{3d}{2})\Gamma(x+3-d)}{\Gamma(x)\Gamma(x+7-2d)} \frac{2}{\Gamma(1-\frac{d}{2})}\,. \tag{5.4}$$

For the expression $T_4(x)$, we can immediately construct a simple first-order difference equation, $x(x+7-2d)T_4(x+1) = (x+5-\frac{3d}{2})(x+3-d)T_4(x)$, from which we get — in complete analogy to Laplace transforming eq. (5.2) – a differential equation for w_4:

$$0 = (d-3)(3d-10-4z)w_4(z) - z(14-5d+4z(d-2))w_4'(z) + 2z^2(1-z)w_4''(z)\,. \tag{5.5}$$

To write its boundary condition eq. (5.4) in a (for numerical treatment) more useful form, note that the behavior of $w_4(z)$ at the singular point $z=1$ is connected to the large-x limit of $T_4(x)$. Using Stirling's formula to write $\frac{\Gamma(x+a)}{\Gamma(x+b)} = x^{a-b}(1 + \frac{(a-b)(a+b-1)}{2x} + \mathcal{O}\left(x^{-2}\right))$, we can fix the three constants in the Ansatz $w_4(z \approx 1) = c_1(1-z)^{c_2}(1+c_3(1-z)+\ldots)$, when comparing $\int_0^1 dz\, z^{x-1} w_4(z \approx 1)$ at large x with $T_4(x \gg 1)$. Hence, writing

$$\begin{aligned} w_4(z) &= c_1(1-z)^{c_2}\frac{2z^{1-\frac{d}{2}}}{d-2}\bar{w}_4(z) = \frac{4(1-z)^{\frac{d}{2}-2}z^{1-\frac{d}{2}}\bar{w}_4(z)}{\Gamma(1-\frac{d}{2})\Gamma(\frac{d}{2}-1)(d-2)} \\ &= \frac{2\sin\frac{\pi d}{2}}{\pi}\, z^{1-\frac{d}{2}}(1-z)^{\frac{d}{2}-2}\bar{w}_4(z)\,, \end{aligned} \tag{5.6}$$

we get simple boundary conditions $\bar{w}_4(1) = \frac{d-2}{2}$, $\bar{w}_4'(1) = \frac{d-2}{2}(\frac{d-2}{2} - c_3) = \frac{(d-4)^2}{2}$ for the new function $\bar{w}_4(z)$, which satisfies the differential equation

$$0 = -(d-4)^2\bar{w}_4(z) + z(10 - 3d + 4z(d-3))\bar{w}_4'(z) + 2z^2(z-1)\bar{w}_4''(z)\,. \tag{5.7}$$

We now get the non-homogeneous solution $v_{NH}(t)$ by varying the constant of the homogeneous solution. Due to the linearity of the differential equation, the full solution is simply the sum of four terms, which when plugged back into the definition of the Laplace transform gives a representation for the master M_2:

$$\begin{aligned} M_2(x) = &\int_0^1 dt\, t^{x+2-d}(1-t)^{\frac{d}{2}-2}(1-2t)^{\frac{d}{2}-2}\int_t^1 dz\, z^{d-4}(1-z)^{1-\frac{d}{2}}(1-2z)^{1-\frac{d}{2}} \times \\ &\times \left\{\sum_{j=1}^{3}\frac{b_j}{\Gamma(1-a_j)}z^{a_j}(1-z)^{-a_j} + w_4(z)\right\}\,. \end{aligned} \tag{5.8}$$

The integral converges (in 4d) for $x > 1$, so one can use it to compute $M_2(2)$ and get $M_2(1)$ via eq. (5.2).

Unable to solve eq. (5.8) for generic d, let us now go to $d = 4-2\epsilon$ dimensions and start expanding. First, we need to solve the differential equation eq. (5.7). Writing $\bar{w}_4(z) = \sum_{n=0}^{\infty}\epsilon^n f_n(z)$, the boundary conditions translate into $f_0(1) = 1$, $f_1(1) = -1$, $f_{n>1}(1) = 0$ and $f_0'(1) = 0$, $f_1'(1) = 0$, $f_2'(1) = 2$, $f_{n>2}'(1) = 0$. The $f_n(z)$ satisfy the differential equations

$$0 = z(z-1)f_n''(z) + (2z-1)f_n'(z) + (3-4z)f_{n-1}'(z) - \frac{2}{z}f_{n-2}(z)\,, \tag{5.9}$$

which have to be solved starting with $n = 0$ (setting $f_{n<0}(z) \equiv 0$).

One can e.g. solve eq. (5.9) in terms of multiple (nested) integrals. In fact, the Ansatz $f_n(z) = \delta_{n,0} - \delta_{n,1} + \int_1^z da\, g_n(a)$ respects the boundary conditions for $f_n(1)$ and transforms eq. (5.9) into a first order differential equation for $g_n(a)$, whose boundary conditions $g_n(1) = 2\delta_{n,2}$ incorporates those for $f_n'(1)$. The homogeneous solution is of the form $g_n^H(a) = \frac{c_n}{a(1-a)}$ and vanishes due to the boundary condition: $c_n \equiv 0$. The inhomogeneous solution now follows by variation of the constant, such that finally

$$f_n(z) = \delta_{n,0} - \delta_{n,1} + \int_1^z \frac{da}{a}\frac{h_n(a)}{1-a}\,, \tag{5.10}$$

$$h_n(a) = 2(\delta_{n,3}-\delta_{n,2})\ln(a) + \int_1^a \frac{db}{b}\frac{(3-4b)h_{n-1}(b)}{1-b} - 2\int_1^a\frac{db}{b}\int_1^b\frac{dc}{c}\frac{h_{n-2}(c)}{1-c}\,. \tag{5.11}$$

The strategy is now clear: $h_n(a) \to f_n(z) \to \bar{w}_4(z) \to w(z) \to M_2(2) \to M_2(1)$. All of these steps can be performed numerically, and there is a discussion of methods in ref. [3].

In practice, we numerically solved for the f_n using Mathematica, changed the order of integrations in eq. (5.8), dealt with the t-integration (semi-) analytically, and finally performed the z-integration numerically. The singular point at $z = 1/2$ was treated as a principal value integral, and the logarithmically divergent regions near $z = 1/2$ and $z = 1$ were split off and treated analytically via a series-expansion in z.

To check the setup, it is possible to start analytically. Solving eq. (5.11), the first couple of orders for h_n read $h_0(a) = 0$, $h_1(a) = 0$, $h_2(a) = -2\ln(a)$ and $h_3(a) = 2\ln(a) - 3\ln^2(a) + 2\mathrm{Li}_2(1-a)$, where $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ is the polylogarithm.

Using eq. (5.10), this then implies $f_0(z) = 1$, $f_1(z) = -1$, $f_2(z) = -\ln^2 z - 2\mathrm{Li}_2(1-z)$, $f_3(z) = (1 + 5\ln(1-z) - \ln(z))\ln^2(z) + 2(1+\ln(z))\mathrm{Li}_2(1-z) + 10\ln(z)\mathrm{Li}_2(z) - 2\mathrm{Li}_3(1-z) - 10\mathrm{Li}_3(z) + 10\zeta_3$.

Knowing now $\bar{w}_4(z) = 1 - \epsilon + \epsilon^2 f_2(z) + \mathcal{O}\left(\epsilon^3\right)$ and using eq. (5.6) to get $w_4(z)$, we can expand the curly bracket of eq. (5.8). The two leading terms cancel, such that we obtain $\{\cdots\} = \frac{2\epsilon^3}{3z}[(z-1)(\pi^2 + 2\ln^2(1-z) - 6\ln z\ln(1-z)) + 3z\ln^2 z + 6\mathrm{Li}_2(1-z)] + \mathcal{O}\left(\epsilon^4\right)$. Now $M_2(x) = \int_0^1 dt\, t^{x-2} \int_t^1 dz\, \frac{\{\cdots\}}{(1-z)(1-2z)} + \mathcal{O}\left(\epsilon^4\right) = \int_0^1 dz\, \frac{\{\cdots\}}{(1-z)(1-2z)} \frac{z^{x-1}}{x-1} + \mathcal{O}\left(\epsilon^4\right)$. We obtain $M_2(2) = 6\zeta_3\epsilon^3 + \mathcal{O}\left(\epsilon^4\right)$, which, using eq. (5.2) at $x = 0$, translates into $M_2(1) = \frac{4}{3(4-d)} M_2(2) = \frac{2}{3\epsilon} M_2(2) = 4\zeta_3\epsilon^2 + \mathcal{O}\left(\epsilon^3\right)$, in nice agreement with the first term of eq. (6.32).

6. Analytic results

For completeness we list here all existing analytic results applicable to our integrals that we are aware of [12]. Additionally, we give the analytic form of some new coefficients of 4-loop master integrals. These were extracted from our high-precision numerical results of section 4, with the help of the integer-relation finding algorithm PSLQ [17] combined with an educated guess of their number content.

Here, we normalize every integral with the appropriate power of the 1-loop tadpole, such that analytic results are independent of the integration measure. Also, recall that we have set $m = 1$.

We will use the following transcendentals:

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n}\,, \tag{6.1}$$

$$a_n = \sum_{k=1}^{\infty} \frac{1}{2^k k^n} = \mathrm{Li}_n\left(\frac{1}{2}\right), \tag{6.2}$$

$$\mathrm{Ls}_j(\theta) = -\int_0^\theta d\tau\, \ln^{j-1}\left|2\sin\frac{\tau}{2}\right|, \tag{6.3}$$

and abbreviate the log-sine integrals at their maximum value as $\mathrm{Ls}_j(\frac{2\pi}{3}) \equiv \mathrm{Ls}_j$ below.

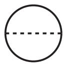
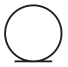
6.1 1-loop

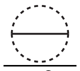
There is one 1-loop topology and one coloring by mass. The 1-loop tadpole is solved in terms of Gamma functions. With measure $\int d^dp$, $J = \int d^dp \frac{1}{p^2+1} = \pi^{d/2}\Gamma(1-d/2)$.

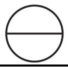
$$\equiv J\,. \tag{6.4}$$

6.2 2-loop

There is one 2-loop topology and three colorings by mass. One of them reduces to simpler cases, while the other two are master integrals. One of the two master integrals has an analytic solution in terms of Gamma functions. The other (fully massive) one can be written in terms of the hypergeometric function ${}_2F_1$ (see eqs. (4.12) and (4.13) in ref. [19]), or alternatively in terms of a one-dimensional integral (see eqs. (21), (15) and (16) in ref. [20]) which has a simple ϵ-expansion (for 4d in terms of log-sine integrals).

 $$= \; -\frac{d-2}{2(d-3)} \Big(\quad \Big)^2 \tag{6.5}$$ 

 $$\frac{\quad}{J^2} = \frac{\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})}{2^{d-2}\Gamma(\frac{1}{2})} \tag{6.6}$$

 $$\frac{\quad}{J^2} = -\frac{3(d-2)}{4(d-3)} \left\{ {}_2F_1\left(\frac{4-d}{2}, 1; \frac{5-d}{2}; \frac{3}{4}\right) - 3^{\frac{d-5}{2}} \frac{2\pi\Gamma(5-d)}{\Gamma(\frac{4-d}{2})\Gamma(\frac{6-d}{2})} \right\} \tag{6.7}$$

$$= -\frac{3(d-2)}{4(d-3)} \left\{ 1 - 3^{\frac{d-3}{2}}(d-4)\int_0^{\frac{\pi}{3}} d\tau (2\sin(\tau))^{4-d} - 3^{\frac{d-5}{2}} \frac{2\pi\Gamma(5-d)}{\Gamma(\frac{4-d}{2})\Gamma(\frac{6-d}{2})} \right\} \tag{6.8}$$

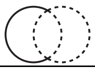
$$\stackrel{d=n-2\epsilon}{=} -\frac{3(n-2-2\epsilon)}{4(n-3-2\epsilon)} \left\{ 1 + 3^{-\epsilon} \frac{\frac{n-4}{2}-\epsilon}{3^{\frac{3-n}{2}}} \sum_{j=0}^{\infty} \frac{(2\epsilon)^j}{j!} \mathrm{Ls}_{j+1}^{(4-n)} - \right.$$
$$\left. -3^{-\epsilon} \frac{3^{\frac{n-5}{2}} 2\pi\Gamma(5-n+2\epsilon)}{\Gamma(\frac{4-n}{2}+\epsilon)\Gamma(\frac{6-n}{2}+\epsilon)} \right\} \tag{6.9}$$

The numbers $\mathrm{Ls}_j^{(a)} = -\int_0^{\frac{2\pi}{3}} d\tau (2\sin\frac{\tau}{2})^a \ln^{j-1}|2\sin\frac{\tau}{2}|$ in the 4d ($n=4$) case are the log-sine integrals $\mathrm{Ls}_j^{(0)} = \mathrm{Ls}_j = \mathrm{Ls}_j(\frac{2\pi}{3})$ of eq. (6.3).

6.3 3-loop

There are three 3-loop topologies.

3-loop, 4 lines: there are four colorings by mass, all of which are master integrals. Two of them have an analytic solution in terms of Gamma functions. The third one (called $D_3(0,1,0,1,1,1)$ in the literature, according to the notation introduced in ref. [15]) can be written in terms of a single hypergeometric function ${}_3F_2$ (see eq. (4.33) of ref. [21], where also the first seven orders of its 4d ϵ-expansion were given in eq. (4.32)).[3] The first seven orders of the 4d ϵ-expansion of the fourth (fully massive) master (called $B_N(0,0,1,1,1,1)$ in the literature) can be deduced from the function B_4 introduced in ref. [24] using the reductions eqs. (6.26) and (6.27) given below. Two more orders could be obtained from B_4 as given in ref. [25], but we refrain from reproducing them here.

 $$\frac{\quad}{J^3} = -\frac{3\Gamma(\frac{6-3d}{2})\Gamma(3-d)\Gamma^2(\frac{d-2}{2})}{\Gamma^3(\frac{2-d}{2})} \tag{6.10}$$

[3] In some sense, the representation in terms of special types of hypergeometric functions can be called an all-order analytic ϵ-expansion, namely when their expansion can be written in terms of rapidly converging (multiple inverse binomial) sums, for which efficient algorithms exist [22]. The 3-loop integrals E_3, D_5 and D_4 [23] below belong to this class as well.

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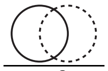
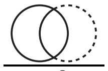
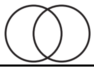


$$\frac{}{J^3} = \frac{2^{d-3}\Gamma(\frac{8-3d}{2})\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{7-2d}{2})\Gamma(\frac{2-d}{2})} \tag{6.11}$$



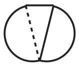
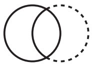
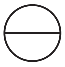
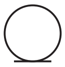
$$\frac{}{J^3} \overset{d=4-2\epsilon}{=} -1-\frac{3}{4}\epsilon+\frac{1}{8}\epsilon^2+\left(\frac{91}{16}-\frac{9}{2}\sqrt{3}\mathrm{Ls}_2\right)\epsilon^3+ \\ +\left(\frac{913}{32}-\frac{3}{4}\sqrt{3}(\pi^3+9(3-2\ln 3)\mathrm{Ls}_2+18\mathrm{Ls}_3)\right)\epsilon^4+ \\ +\left(\frac{7027}{64}+\frac{1}{8}\sqrt{3}(9\pi^3(2\ln 3-3)+64\mathrm{Ls}_4\left(\frac{\pi}{3}\right)-9(67-54\ln 3+18\ln^2 3)\mathrm{Ls}_2+\right. \\ \left.+162(2\ln 3-3)\mathrm{Ls}_3-216\mathrm{Ls}_4+184\pi\zeta_3)\right)\epsilon^5+ \\ +\left(\frac{48601}{128}+\right. \\ +\sqrt{3}\left(-\frac{3}{16}\pi^3(67+18\ln 3(\ln 3-3))-\frac{23}{36}\pi^5+\frac{69}{2}\pi(3-2\ln 3)\zeta_3+\right. \\ +81\pi\mathrm{Ls}_4'+\frac{9}{16}(-457+6\ln 3(67+3\ln 3(2\ln 3-9)))\mathrm{Ls}_2- \\ -\frac{27}{8}(67+8\pi^2+18\ln 3(\ln 3-3))\mathrm{Ls}_3+\frac{81}{2}(2\ln 3-3)\mathrm{Ls}_4- \\ -\frac{81}{2}\mathrm{Ls}_5+12(3-2\ln 3)\mathrm{Ls}_4\left(\frac{\pi}{3}\right)+ \\ \left.\left.+49\mathrm{Ls}_5\left(\frac{\pi}{3}\right)-\frac{243}{4}\mathrm{Ls}_5'\right)\right)\epsilon^6+\mathcal{O}\left(\epsilon^7\right) \tag{6.12}$$

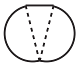
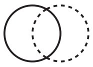
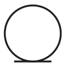


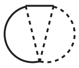
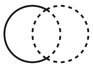
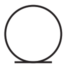
$$\frac{}{J^3} \overset{d=4-2\epsilon}{=} -2-\frac{5}{3}\epsilon-\frac{1}{2}\epsilon^2+\frac{103}{12}\epsilon^3+\frac{7}{24}(163-128\zeta_3)\epsilon^4+ \\ +\left(\frac{9055}{48}+\frac{136\pi^4}{45}+\frac{32}{3}\ln^2 2(\pi^2-\ln^2 2)-168\zeta_3-256a_4\right)\epsilon^5+ \\ +\left(\frac{63517}{96}+\frac{16}{5}\ln^4 2(4\ln 2-15)-\frac{16}{3}\pi^2\ln^2 2(4\ln 2-9)-\frac{68}{15}\pi^4(4\ln 2-3)-\right. \\ \left.-\frac{1876}{3}\zeta_3+1240\zeta_5-1152a_4-1536a_5\right)\epsilon^6+\mathcal{O}\left(\epsilon^7\right) \tag{6.13}$$

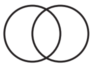
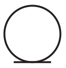
$\mathrm{Ls}_j' = -\int_0^{\frac{2\pi}{3}} d\tau\tau^{j-3}\ln^2|2\sin\frac{\tau}{2}|$ are special values of the generalized log-sine function [21].

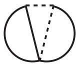
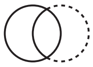
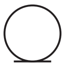
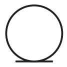
3-loop, 5 lines: there are eleven colorings by mass. Eight of them reduce, while the remaining three are master integrals. One of the masters has an analytic solution in terms of Gamma functions. The second one (called E_3 in the literature) can be written in terms of the hypergeometric function ${}_2F_1$, cf. eq. (4.24) of ref. [21]. Its first six terms of the 4d ϵ-expansion (we will only reproduce the first five of them below) are given in eqs. (4.16), (4.18) of ref. [21]. The first five terms of the 4d ϵ-expansion of the third (fully massive) master integral (called D_5 in the literature) can be deduced from ref. [26] using the reduction given in eq. (6.25) below. One more term has recently been given in eq. (3.28) of ref. [27], but we refrain from listing it here.

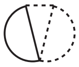
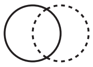
 $= \frac{1}{6(d-3)}\Big\{(3d-8)$  $-\,3(d-2)$   $\Big\}$ (6.14)

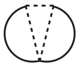
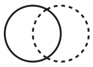
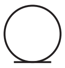
 $= \frac{1}{4(d-3)}\Big\{(3d-8)$  $+\,\frac{(d-2)^2}{d-3}\Big($  $\Big)^3\Big\}$ (6.15)

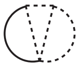
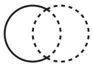
 $= \frac{1}{2(d-3)}\Big\{(3d-8)$  $-\,(d-2)$   $\Big\}$ (6.16)

 $= -\frac{1}{4(d-4)}\Big\{(3d-8)$  $+\,\frac{2(d-2)^2}{d-3}\Big($  $\Big)^3\Big\}$ (6.17)

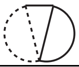
 $= \frac{3d-8}{2}$  $-\,\frac{d-2}{2}$   $+\,\frac{(d-2)^2}{2(d-3)}\Big($  $\Big)^3$ (6.18)

 $= -\frac{3d-8}{4(2d-7)}$  (6.19)

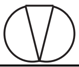
 $= \frac{3d-8}{d-2}$  $-\,2$   (6.20)

 $= -\frac{3d-8}{d-4}$  (6.21)

 $/\,J^3 = \frac{\pi^3}{\sin^2(\frac{\pi d}{2})\sin(\frac{3\pi d}{2})}\frac{\Gamma^2(\frac{d-2}{2})}{\Gamma^3(\frac{2-d}{2})\Gamma^2(d-2)\Gamma(\frac{d}{2})}$ (6.22)

 $/\,J^3$

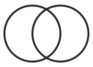
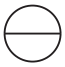
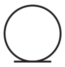
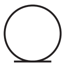
$$\overset{d=4-2\epsilon}{=} \frac{2}{3}+\frac{5}{3}\epsilon+\left(5+\frac{\pi^2}{6}-3\sqrt{3}\mathrm{Ls}_2\right)\epsilon^2+ \\ +\left(\frac{44}{3}+\frac{\pi^2}{3}+\frac{1}{3}\zeta_3-3\sqrt{3}\left(\frac{5\pi^3}{162}+(2-\ln 3)\mathrm{Ls}_2+\mathrm{Ls}_3\right)\right)\epsilon^3+ \\ +\left(\frac{128}{3}+\frac{5\pi^2}{6}-\frac{\pi^4}{60}+\frac{10}{3}\zeta_3+ \right. \\ +\sqrt{3}\left(-\frac{1}{6}(2\pi^2+9\ln 3(10+(\ln 3-4)\ln 3))\mathrm{Ls}_2+3(\ln 3-2)\mathrm{Ls}_3- \right. \\ \left.\left.-2\mathrm{Ls}_4-\frac{80}{27}\mathrm{Ls}_4\Big(\frac{\pi}{3}\Big)+\frac{5\pi^3}{54}(\ln 3-2)+\frac{94}{27}\pi\zeta_3\right)\right)\epsilon^4+\mathcal{O}\left(\epsilon^5\right) \tag{6.23}$$

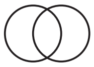
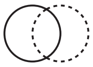
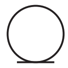
 $/\,J^3$

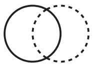
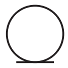
$$\overset{d=4-2\epsilon}{=} 1+\frac{8}{3}\epsilon+\left(\frac{25}{3}-6\sqrt{3}\mathrm{Ls}_2\right)\epsilon^2+ \\ +\left(\frac{76}{3}-6\zeta_3+\sqrt{3}\left(-\frac{\pi^3}{3}+6(\ln 3-2)\mathrm{Ls}_2-6\mathrm{Ls}_3\right)\right)\epsilon^3+ \\ +\left(76-\frac{7\pi^4}{10}+18\mathrm{Ls}_2^2-12\pi\mathrm{Ls}_3+18\mathrm{Ls}'_4+\left(-\frac{92}{3}+4\sqrt{3}\pi+26\ln 3\right)\zeta_3+ \right. \\ +\sqrt{3}\left(\frac{\pi^3}{3}(\ln 3-2)-3(10-4\ln 3+\ln^2 3)\mathrm{Ls}_2+ \right. \\ \left.\left.+6(\ln 3-2)\mathrm{Ls}_3-4\mathrm{Ls}_4\right)\right)\epsilon^4+\mathcal{O}\left(\epsilon^5\right) \tag{6.24}$$

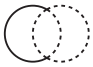
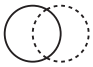
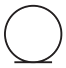
3-loop, 6 lines: there are ten colorings by mass. The first two terms of their 4d ϵ-

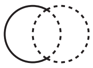
expansion are given in ref. [26]. Five of them reduce, while the remaining five are masters. The third term of the 4d ϵ-expansion of one of the masters (called D_4 in the literature) is given in eq. (4.10) of ref. [21].[4] The remaining four masters (called D_M, D_N, D_3 and D_6, respectively) are read from ref. [26].

 $= -\frac{2(d-3)}{3(d-4)}$  $+\frac{3d-8}{12(d-4)}$  $-\frac{2(d-2)}{3(d-4)}$   $-$
$-\frac{(d-2)^2}{6(d-4)(d-3)}\Big($  $\Big)^3$ (6.25)

 $= \frac{(3d-10)(3d-8)}{16(d-4)^2}\Big($  $+\frac{4(d-4)}{2d-7}$  $\Big)+$
$+\frac{(d-2)^2(5d-18)}{8(d-4)^2(d-3)}\Big($  $\Big)^3$ (6.26)

 $= -\frac{3(3d-10)(3d-8)}{16(d-4)(2d-7)}$  $-\frac{(d-2)^2}{8(d-4)(d-3)}\Big($  $\Big)^3$ (6.27)

 $= -\frac{(3d-10)(3d-8)}{(d-4)^2}\Big($  $+\frac{d-4}{4(2d-7)}$  $\Big)+$
$+\frac{d-2}{d-4}$   (6.28)

 $= \frac{2(d-3)}{d-4}$  $+\frac{2(3d-10)(3d-8)}{(d-4)^2}$  (6.29)

 $/J^3 \overset{d=4-2\epsilon}{=} -2\zeta_3\epsilon^2+\left(\frac{77\pi^4}{1080}+\frac{27}{2}\mathrm{Ls}_2^2\right)\epsilon^3+ +\left(-\frac{21}{8}\chi_5+\frac{161}{54}\pi\mathrm{Ls}_4\left(\frac{\pi}{3}\right)-\frac{367}{216}\pi^3\mathrm{Ls}_2- -7\pi\mathrm{Ls}_4-2\zeta_3+\frac{2615}{432}\pi^2\zeta_3-\frac{2047}{216}\zeta_5\right)\epsilon^4+\mathcal{O}\left(\epsilon^5\right)$ (6.30)

 $/J^3 \overset{d=4-2\epsilon}{=} -2\zeta_3\epsilon^2+\left(\frac{11\pi^4}{180}+9\mathrm{Ls}_2^2\right)\epsilon^3+\mathcal{O}\left(\epsilon^4\right)$ (6.31)

 $/J^3 \overset{d=4-2\epsilon}{=} -2\zeta_3\epsilon^2+\left(\frac{7\pi^4}{60}+\frac{2}{3}\ln^2 2(\pi^2-\ln^2 2)-16a_4\right)\epsilon^3+\mathcal{O}\left(\epsilon^4\right)$ (6.32)

 $/J^3 \overset{d=4-2\epsilon}{=} -2\zeta_3\epsilon^2+\left(\frac{\pi^4}{24}+\frac{27}{2}\mathrm{Ls}_2^2\right)\epsilon^3+\mathcal{O}\left(\epsilon^4\right)$ (6.33)

 $/J^3 \overset{d=4-2\epsilon}{=} -2\zeta_3\epsilon^2+\left(\frac{17\pi^4}{90}+\frac{2}{3}\ln^2 2(\pi^2-\ln^2 2)+9\mathrm{Ls}_2^2-16a_4\right)\epsilon^3+\mathcal{O}\left(\epsilon^4\right)$ (6.34)

$\chi_5=\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}\frac{1}{n^2}\left(\sum_{j=1}^{n-1}\frac{1}{j}\right)^3\approx 0.0678269619272\ldots$ is a special case of a binomial sum [21].

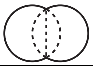
[4] Note that there is a typo in eq. (4.10) of ref. [21]. The second-last term should read $-\frac{161}{54}\pi\mathrm{Ls}_4(\frac{\pi}{3})$, see also ref. [27].

6.4 4-loop

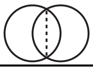
There are ten topologies.

4-loop QED-type cases, 5 lines: there is one topology, BB.

There are two QED-type colorings of BB. Both of them are masters. One is known analytically in terms of Gamma functions, while the other one is new.[5] Interestingly, the analytic value of the last two terms in eq. (6.36) were obtained in a physics computation in which this master integral contributed [29].



$$\frac{\quad}{J^4} = 3(d-2)4^{d-3}\frac{\Gamma(5-2d)\Gamma(\frac{8-3d}{2})\Gamma(\frac{5-d}{2})\Gamma^2(\frac{d}{2})}{\Gamma(\frac{11-3d}{2})\Gamma^3(\frac{4-d}{2})} \tag{6.35}$$



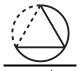
$$\frac{\quad}{J^4} \overset{d=4-2\epsilon}{=} -1-\frac{1}{2}\epsilon+\frac{17}{36}\epsilon^2+\frac{1}{216}\epsilon^3-\frac{37207}{1296}\epsilon^4+\left(-\frac{1976975}{7776}+\frac{1792}{9}\zeta_3\right)\epsilon^5+ \\ +\left(-\frac{72443143}{46656}-\frac{4352\pi^4}{135}+\frac{1024}{9}\ln^2 2(\ln^2 2-\pi^2)+\frac{8192}{3}a_4+\frac{47488}{27}\zeta_3\right)\epsilon^6+ \\ +\mathcal{O}\left(\epsilon^7\right) \tag{6.36}$$

4-loop QED-type cases, 6 lines: there are two topologies, T and G.

There are four QED-type colorings of T. All of them are masters. One of them is known analytically, while the first six orders of the 4d ϵ-expansion of two others were given in eq. (16) of ref. [7] and eq. (18) of ref. [11],[6] respectively. The fourth one is new.

$$\frac{\quad}{J^4} = \frac{8^{d-3}\Gamma^3(\frac{1}{2})\Gamma(6-2d)\Gamma^3(\frac{d}{2})}{\sin(\frac{3\pi d}{2})\Gamma(\frac{11-3d}{2})\Gamma^2(\frac{4-d}{2})\Gamma^2(d-2)} \tag{6.37}$$

$$\frac{\quad}{J^4} \overset{d=4-2\epsilon}{=} \frac{3}{2}+\frac{7}{2}\epsilon+\frac{9}{2}\epsilon^2+\left(-\frac{39}{2}-3\zeta_3\right)\epsilon^3+\left(-208-\frac{\pi^4}{20}+109\zeta_3\right)\epsilon^4+ \\ +\left(-1254-\frac{547\pi^4}{60}+32\ln^2 2(\ln^2 2-\pi^2)+768a_4+855\zeta_3+189\zeta_5\right)\epsilon^5+ \\ +\mathcal{O}\left(\epsilon^6\right) \tag{6.38}$$



$$\frac{\quad}{J^4} \overset{d=4-2\epsilon}{=} \frac{2}{3}+\frac{4}{3}\epsilon+\frac{2}{3}\epsilon^2+\frac{4}{3}(-11+4\zeta_3)\epsilon^3+\left(-116-\frac{4\pi^4}{15}+\frac{200\zeta_3}{3}\right)\epsilon^4+ \\ +\left(-\frac{1928}{3}-\frac{326\pi^4}{45}+\frac{64}{3}\ln^2 2(\ln^2 2-\pi^2)+512a_4+\frac{1192\zeta_3}{3}+96\zeta_5\right)\epsilon^5+ \\ +\mathcal{O}\left(\epsilon^6\right) \tag{6.39}$$

$$\frac{\quad}{J^4} \overset{d=4-2\epsilon}{=} \frac{1}{4}+\frac{1}{2}\epsilon+0\cdot\epsilon^2+\left(-8+\frac{13}{2}\zeta_3\right)\epsilon^3+\left(-\frac{241}{4}-\frac{5\pi^4}{8}+4\zeta_3\right)\epsilon^4+$$

[5] This integral has also been expanded in terms of 1-dimensional harmonic polylogarithms [28].

[6] Note that in ref. [11] the last term of eq. (6.39) involves a numerical coefficient $N_{10} \approx 5.3111546$, which we have determined to be $N_{10} = \frac{49\pi^4}{720} + \frac{1}{6}\ln^2 2\left(\pi^2 - \ln^2 2\right) - 4a_4$, using our high-precision result eq. (4.10) and PSLQ [17].

$$+\left(-\frac{669}{2}-\frac{\pi^4}{5}-36\zeta_3+\frac{693}{2}\zeta_5\right)\epsilon^5+\mathcal{O}\left(\epsilon^6\right) \tag{6.40}$$

There are five QED-type colorings of G. All of them reduce.

$$= \frac{2d-5}{4(d-3)} \quad - \frac{d-2}{2(d-3)} \tag{6.41}$$

$$= \frac{2d-5}{4} \quad - \frac{d-2}{2} \quad + \frac{(d-2)^3}{8(d-3)^2}\left(\quad\right)^4 \tag{6.42}$$

$$= -\frac{2d-5}{6(d-3)} \tag{6.43}$$

$$= -\frac{(2d-5)(3d-8)}{6(d-3)(d-4)} \tag{6.44}$$

$$= \frac{2d-5}{2(d-3)} \quad - \frac{d-2}{2(d-3)} \tag{6.45}$$

4-loop QED-type cases, 7 lines: there are three topologies, VB, N and U.

There are seven QED-type colorings of VB. Five of them reduce. There are two master integrals. Both are new.

$$= \frac{(2d-5)(3d-10)(3d-8)}{3(d-4)^2(3d-11)} \quad + \frac{2(d-3)^2}{(d-4)(3d-11)} \tag{6.46}$$

$$= -\frac{(2d-5)(3d-8)}{3(d-4)^2} \quad - \frac{4(d-3)^2}{3(d-4)(3d-10)} \quad + \\ + \frac{(d-2)(3d-8)}{8(2d-7)(3d-10)} \tag{6.47}$$

$$= \frac{(2d{-}5)(3d{-}8)}{3(d{-}4)(d{-}3)} \quad - \frac{d{-}3}{d{-}4} \quad - \frac{(d{-}2)(3d{-}8)}{4(d{-}4)(d{-}3)} \tag{6.48}$$

$$= \frac{(2d{-}5)(3d{-}8)}{16(d{-}4)(d{-}3)} \quad - \frac{2(d{-}3)}{3(d{-}4)} \quad - \frac{(d{-}2)(3d{-}8)}{8(d{-}4)(d{-}3)} \quad - \\ - \frac{(d-2)^3}{32(d-4)(d-3)^2}\left(\quad\right)^4 \tag{6.49}$$

$$= \frac{2}{3d{-}10}\left\{(d{-}3) \quad - \frac{3d-8}{2(d{-}3)}\left(\frac{2d{-}5}{3} \quad - \frac{d{-}2}{4} \quad \right)\right\} \tag{6.50}$$

$$\frac{\quad}{J^4} \overset{d=4-2\epsilon}{=} -\frac{1}{6}-\frac{5}{6}\epsilon-\left(\frac{11}{3}+\zeta_3\right)\epsilon^2+\left(-\frac{44}{3}-\frac{\pi^4}{60}+\frac{2}{3}\zeta_3\right)\epsilon^3+ \\ +\left(-\frac{332}{6}-\frac{\pi^4}{6}+\frac{31}{3}\zeta_3+53\zeta_5\right)\epsilon^4+\mathcal{O}\left(\epsilon^5\right) \tag{6.51}$$

$$\frac{\quad}{J^4} \overset{d=4-2\epsilon}{=} -\frac{1}{6}-\frac{5}{6}\epsilon-\left(\frac{11}{3}+\frac{1}{2}\zeta_3\right)\epsilon^2+\left(-\frac{44}{3}-\frac{\pi^4}{120}+\frac{13}{6}\zeta_3\right)\epsilon^3+ \\ +\left(-\frac{166}{3}-\frac{5\pi^4}{24}+\frac{29}{6}\zeta_3+\frac{43}{2}\zeta_5\right)\epsilon^4+\mathcal{O}\left(\epsilon^5\right) \tag{6.52}$$

There are five QED-type colorings of N. All of them reduce.

$$= \frac{(2d-5)(3d-8)}{6(d-3)^2} \quad - \frac{(d-2)(3d-8)}{4(d-3)^2} \quad -$$

$$-\frac{(d-2)^3}{8(d-3)^3}\Big(\quad\Big)^4 \tag{6.53}$$

$$= -\frac{(2d-5)(3d-8)}{6(d-4)(d-3)} + \frac{(d-2)(3d-8)}{8(d-3)(2d-7)} \tag{6.54}$$

$$= \frac{(2d-5)(3d-8)}{6(d-4)(3d-11)} \tag{6.55}$$

$$= -\frac{(2d-5)(3d-8)}{16(d-4)} + \frac{(d-2)(3d-8)}{4(2d-7)} -$$

$$-\frac{3(d-2)^3}{32(d-4)(d-3)}\Big(\quad\Big)^4 \tag{6.56}$$

$$= \frac{(2d-5)(3d-8)}{16(d-3)} + \frac{(d-2)(3d-8)}{8(d-4)(d-3)} -$$

$$-\frac{(d-2)(3d-8)}{8(d-3)} + \frac{(d-2)^3(3d-4)}{32(d-4)(d-3)^2}\Big(\quad\Big)^4 \tag{6.57}$$

There are four QED-type colorings of U. All of them reduce.

$$= \frac{2}{3} - \frac{(d-2)(3d-8)}{8(d-3)^2} - \frac{(d-2)^3}{8(d-3)^3}\Big(\quad\Big)^4 \tag{6.58}$$

$$= -\frac{d-3}{d-5} - \frac{3(d-2)(3d-8)}{8(d-5)(d-4)} - \frac{3(d-2)^3}{4(d-5)(d-4)(d-3)}\Big(\quad\Big)^4 \tag{6.59}$$

$$= -\frac{d-3}{2(d-4)} - \frac{(d-2)(3d-8)}{8(d-4)(2d-7)} \tag{6.60}$$

$$= -\frac{d-3}{3d-11} \tag{6.61}$$

4-loop QED-type cases, 8 lines: there are two topologies, VV and W.

There are seven QED-type colorings of VV. All of them reduce.

$$= -\frac{(2d-7)(2d-5)(3d-10)(3d-8)}{6(d-4)^2(3d-13)(3d-11)} - \frac{(d-3)^2(2d-7)}{(d-4)(3d-13)(3d-11)} \tag{6.62}$$

$$= \frac{(2d-7)(2d-5)(3d-10)(3d-8)}{18(d-4)^2(d-3)(3d-11)} - \frac{(d-3)(2d-7)}{3(d-4)(3d-11)} -$$

$$-\frac{(d-2)(3d-10)(3d-8)}{48(d-4)(d-3)(2d-7)} \tag{6.63}$$

$$= -\frac{(2d-7)(2d-5)(3d-10)(3d-8)}{6(d-4)^2(d-3)(3d-11)} + \frac{(d-3)(2d-7)}{4(d-4)^2} +$$

$$+\frac{(d-2)(3d-8)(5d^2-35d+61)}{16(d-4)^2(d-3)(2d-7)} \tag{6.64}$$

$$= -\frac{(2d-7)(2d-5)(3d-10)(3d-8)}{6(d-4)^2(d-3)(3d-11)} - \frac{4(d-3)^2(2d-7)}{3(d-4)(3d-11)(3d-10)} +$$

$$+\frac{(d-2)(3d-8)(95d^3-989d^2+3428d-3956)}{32(d-4)(d-3)(2d-7)(3d-11)(3d-10)} +$$

$$+\frac{(d-2)^3}{16(d-4)(d-3)^2}\Big(\quad\Big)^4 \tag{6.65}$$

$$
= -\frac{(2d-5)(3d-11)(3d-8)}{32(d-4)(2d-9)} - \frac{(d-3)^2}{4(d-4)(2d-9)} - \frac{(3d-10)}{4(2d-9)} + \frac{(d-2)(3d-8)(12d^2-101d+204)}{32(d-4)(2d-9)(2d-7)} - \frac{(d-2)(3d-8)}{16(d-4)(2d-9)} - \frac{3(d-2)^3(3d-7)}{64(d-4)(d-3)(2d-9)} \left(\quad \right)^4 \qquad (6.66)
$$

$$
= \frac{(2d-5)(3d-8)}{12(d-3)(2d-7)} + \frac{(2d-5)(3d-8)}{64(d-3)(2d-7)} - \frac{2(d-3)^2}{3(2d-7)(3d-10)} + \frac{3d-10}{2(2d-7)} - \frac{(d-2)^3(73d^2-512d+896)}{128(d-4)^2(d-3)^2(2d-7)} \left(\quad \right)^4 - \frac{(d-2)(3d-8)(19d^2-128d+216)}{(16(d-4)(d-3)(2d-7)(3d-10)} - \qquad (6.67)
$$

$$
- \frac{(d-2)(3d-10)(3d-8)}{32(d-4)^2(d-3)} - \frac{(d-2)^3(73d^2-512d+896)}{128(d-4)^2(d-3)^2(2d-7)} \left(\quad \right)^4
$$

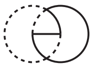
$$
= \frac{(2d-7)(2d-5)(3d-10)(3d-8)}{64(d-4)^2(d-3)} + \frac{(d-3)(2d-7)}{3(d-5)(d-4)} - \frac{(d-2)(3d-10)(3d-8)}{8(d-4)(2d-7)} + \frac{(d-2)(3d-8)(11d^2-77d+134)}{32(d-5)(d-4)^2(d-3)} + \frac{(d-2)^3(18d^3-129d^2+245d-58}{128(d-5)(d-4)^2(d-3)^2} \left(\quad \right)^4 \qquad (6.68)
$$

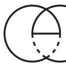
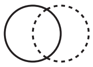
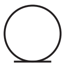
There are five QED-type colorings of W. Four of them reduce. There is one master integral, whose leading term in 4d can be read from ref. [30] due to the fact that it does not contain any infrared divergences.

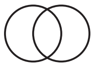
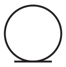
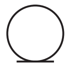
$$
= -\frac{(2d-5)(3d-8)(27d^3-283d^2+990d-1156)}{12(d-4)^3(2d-7)(3d-11)} - \frac{(d-3)(3d-10)}{2(d-4)(2d-7)} - \frac{2(d-3)^4(5d-18)}{3(d-4)^2(2d-7)(3d-11)(3d-10)} - \frac{(2d-5)(3d-8)}{64(d-4)(2d-7)} + \frac{(d-2)(3d-8)(5d-18)}{16(d-4)(3d-11)(3d-10)} - \frac{7(d-2)^3}{128(d-4)(d-3)(2d-7)} \left(\quad \right)^4 \qquad (6.69)
$$

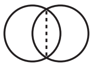
$$
= \frac{2(2d-7)(2d-5)(3d-10)(3d-8)}{9(d-4)^3(3d-11)} + \frac{8(d-3)^3(2d-7)}{9(d-4)^2(3d-11)(3d-10)} + \frac{2(d-3)^3(2d-7)}{3(d-4)^2(3d-11)} - \frac{(d-2)(3d-8)(7d^2-48d+82)}{24(d-4)(2d-7)(3d-11)(3d-10)} \qquad (6.70)
$$

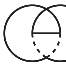
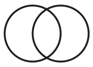
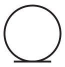
$$
= \frac{(2d-5)(3d-8)}{6(d-4)(2d-7)} - \frac{(2d-5)(3d-8)}{32(d-3)(2d-7)} + \frac{d-3}{2(d-4)} -
$$

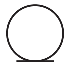
$-\dfrac{4(d-3)^3}{3(d-4)(2d-7)(3d-10)}$  $+\dfrac{(d-3)(3d-10)}{(d-4)(2d-7)}$  $-$

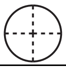
$-\dfrac{3d-10}{2(d-4)}$  $-\dfrac{(d-2)(3d-8)}{8(2d-7)(3d-10)}$   $+$ (6.71)

$+\dfrac{(d-2)(3d-8)}{16(d-4)(d-3)}$   $+\dfrac{(d-2)^3(9d-28)}{64(d-4)(d-3)^2(2d-7)}\Big($  $\Big)^4$

 $=\dfrac{(2d-5)(3d-8)}{32(d-4)(d-3)}$  $+\dfrac{2(d-3)^2(2d-7)}{3(d-4)^2(3d-11)}$  $-\dfrac{d-3}{2(d-4)}$  $+$

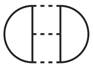
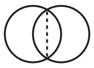
$+\dfrac{3d-10}{2(d-4)}$  $+\dfrac{(d-2)(3d-8)(d^2-4d+2)}{8(d-4)^2(d-3)(3d-11)}$   $+$

$+\dfrac{(d-2)^3(29d^2-177d+268)}{64(d-4)^2(d-3)^2(3d-11)}\Big($  $\Big)^4$ (6.72)

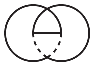
 $\dfrac{}{J^4}\overset{d=4-2\epsilon}{=}5\zeta_5\epsilon^3+\mathcal{O}\left(\epsilon^4\right)$ (6.73)

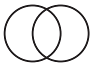
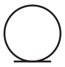
4-loop QED-type cases, 9 lines: there are two topologies, H and X.

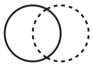
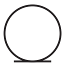
There are five QED-type colorings of H. All of them reduce.

 $=-\dfrac{3(2d-5)(3d-11)(3d-8)(54-29d+4d^2)}{256(d-5)(d-4)^2(d-3)(2d-9)}$  $-$

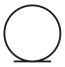
$-\dfrac{4(d-3)^2(2d-7)}{3(d-5)(3d-13)(3d-11)}$  $+$

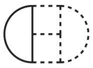
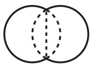
$+\dfrac{3(d-3)(d^2-11d+27)}{8(d-5)(d-4)(2d-9)}$  $-\dfrac{9(d-4)(3d-10)}{8(d-5)(2d-9)}$  $-$

$-\dfrac{(d-2)(3d-8)(13d^3-77d^2+9d+351)}{32(d-5)(d-3)(2d-9)(3d-13)(3d-11)}$   $-$

$-\dfrac{3(d-2)(3d-8)(13d^2-92d+162)}{64(d-5)(d-4)(2d-9)(2d-7)}$   $-$

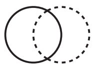
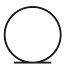
$-\dfrac{(d-2)^3(2452d^5-43031d^4+329345d^3-1198763d^2+2170827d-1564110)}{512(d-5)(d-4)^2(d-3)^2(2d-9)(3d-13)(3d-11)}\times$

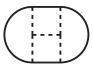
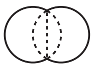
$\times\Big($  $\Big)^4$ (6.74)

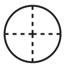
 $=-\dfrac{3(2d-7)(2d-5)(3d-10)(3d-8)}{2(d-4)(3d-14)(3d-13)(3d-11)}$  $-$

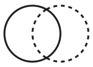
$-\dfrac{6(d-3)^2(2d-7)}{(3d-14)(3d-13)(3d-11)}$  $-$

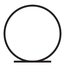
$-\dfrac{32(d-3)^3(2d-7)}{9(3d-14)(3d-13)(3d-11)(3d-10)}$  $+$

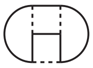
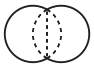
$+\dfrac{(d-2)(3d-8)(139d^3-1495d^2+5344d-6348)}{96(2d-7)(3d-14)(3d-13)(3d-11)(3d-10)}$   (6.75)

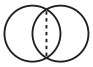
 $=\dfrac{7(2d-7)(2d-5)(3d-10)(3d-8)}{9(d-4)^2(3d-13)(3d-11)}$  $+\dfrac{4(d-3)^2(2d-7)}{3(d-4)(3d-13)(3d-11)}$  $+$

$$+\frac{32(d-3)^3(2d-7)}{9(d-4)(3d-13)(3d-11)(3d-10)}$$  $$-\frac{d-4}{2(3d-13)}$$  $$-$$

$$-\frac{(d-2)(3d-8)(409d^3-4285d^2+14944d-17348)}{96(d-4)(2d-7)(3d-13)(3d-11)(3d-10)}$$   $$-$$

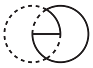
$$-\frac{(d-2)^3}{16(d-4)(d-3)(3d-13)}\Big($$  $$\Big)^4 \qquad (6.76)$$

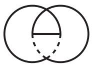
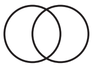
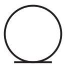
 $$=-\frac{(2d-5)(3d-8)(3d^3-19d^2+19d+38)}{12(d-5)(d-4)^2(2d-7)(3d-11)}$$  $$+$$

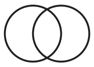
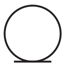
$$+\frac{(2d-5)(3d-11)(3d-8)(2d^3-14d^2+23d+6)}{128(d-5)(d-4)^3(d-3)(2d-7)}$$  $$-$$

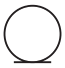
$$-\frac{2(d-3)^2(2d-7)}{3(d-5)(d-4)(3d-11)}$$  $$-$$

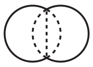
$$-\frac{2(d-3)^3(17d^2-125d+230)}{3(d-5)(d-4)(2d-7)(3d-11)(3d-10)}$$  $$+$$

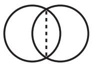
$$+\frac{(d-3)(5d^2-37d+69)}{4(d-5)(d-4)^2}$$  $$+\frac{3(d-3)(3d-10)}{2(d-5)(2d-7)}$$  $$- \qquad (6.77)$$

$$-\frac{3(3d-10)}{4(d-5)}$$  $$-\frac{(d-2)(3d-8)(10d^3-87d^2+235d-186)}{32(d-5)(d-4)^2(d-3)(3d-11)}$$   $$-$$

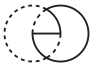
$$-\frac{(d-2)(3d-8)(38d^4-568d^3+3174d^2-7863d+7290)}{16(d-5)(d-4)^2(2d-7)(3d-11)(3d-10)}$$   $$-$$

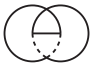
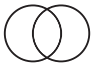
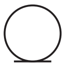
$$-\frac{(d-2)^3(418d^5-7346d^4+51389d^3-178846d^2+309603d-213234)}{256(d-5)(d-4)^3(d-3)^2(2d-7)(3d-11)}\Big($$  $$\Big)^4$$

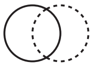
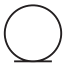
 $$=-\frac{(2d-5)(3d-8)(69d^3-725d^2-2543d-2978)}{12(d-4)^2(2d-9)(2d-7)(3d-11)}$$  $$-$$

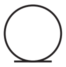
$$-\frac{3(2d-5)(3d-8)}{128(d-3)(2d-9)(2d-7)}$$  $$-$$

$$-\frac{2(d-3)^3(d-2)(13d-47)}{9(d-4)(2d-9)(2d-7)(3d-11)(3d-10)}$$  $$-$$

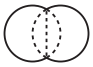
$$-\frac{(d-3)(d^2-11d+27)}{8(d-4)^2(2d-9)}$$  $$-\frac{3(d-3)(3d-10)}{2(2d-9)(2d-7)}$$  $$+$$

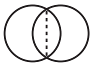
$$+\frac{3(3d-10)}{8(2d-9)}$$  $$+\frac{(d-2)(3d-8)(5d-18)}{64(d-4)^2(d-3)(2d-9)}$$   $$+$$

$$+\frac{(d-2)(3d-8)(1139d^4-16453d^3+89068d^2-214178+193044)}{192(d-4)^2(2d-9)(2d-7)(3d-11)(3d-10)}$$   $$+$$

$$+\frac{(d-2)^3(48d^3-445d^2+1352d-1344)}{256(d-4)^2(d-3)^2(2d-9)(2d-7)}\Big($$  $$\Big)^4 \qquad (6.78)$$

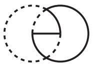
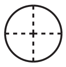
There are two QED-type colorings of X. One reduces. The other one is a master integral. Its leading coefficient (denoted by X_0 below) is not yet known analytically.

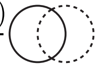
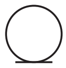
 $$=\frac{(2d-5)(3d-8)(2109d^4-31288d^3+173302d^2-425005d+389562)}{36(d-4)(2d-9)^2(2d-7)(3d-13)(3d-11)}$$  $$+$$

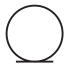
$$+\frac{(2d-5)(3d-8)(24d^3-268d^2+1003d-1258)}{256(d-4)^2(2d-9)^2(2d-7)}$$  $$+$$

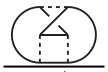
$$+\frac{14(d-3)^2(2d-7)}{3(2d-9)(3d-13)(3d-11)}$$  $+$

$$+\frac{2(d-3)^3(295d^3-3332d^2+12431d-15334)}{9(2d-9)^2(2d-7)(3d-13)(3d-11)(3d-10)}$$  $+$

$$+\frac{3(d-4)(d-3)(3d-10)}{2(2d-9)^2(2d-7)}$$  $+\frac{(d-4)^2}{2(2d-9)(3d-13)}$  $-$

$$-\frac{(d-2)(3d-8)(599d^4-9067d^3+51340d^2-12886d+121044)}{48(2d-9)^2(2d-7)(3d-13)(3d-11)(3d-10)}$$   $+$

$$+\frac{(d-2)^3(392d^4-6204d^3+36843d^2-97323d+96502)}{512(d-4)^2(d-3)(2d-9)^2(2d-7)(3d-13)}\Big($$  $\Big)^4$ (6.79)



$$\frac{\quad}{J^4} \overset{d=4-2\epsilon}{=} X_0\epsilon^4+\mathcal{O}\left(\epsilon^5\right) \tag{6.80}$$

All the above formulas agree with our numerical results of section 4.

7. Conclusions

We have employed the general method of numerically solving single-scale integrals in terms of their ϵ-expansion around $d = 4 - 2\epsilon$ via difference equations, to high precision and to high ϵ-orders. We have covered the set of all vacuum master integrals up to three loops, as well as 'QED-type' vacuum master integrals at 4-loop order. These integrals play a role in state-of-the-art perturbative calculations for precision tests of the standard model.

The main vehicle of solving the difference equations treated in this work was a formal representation in terms of factorial series, which could then be evaluated numerically in a truncated form.

In cases where the factorial series representation does not converge, a more general (and hence more complicated) method can be used, which transforms the problem into differential equations. We have encountered only one such case, and have shown in detail how it can be represented in terms of multiple integrals, which we then solved numerically.

Furthermore, we have made an attempt to collect all presently known analytic results for the class of vacuum master integrals that we have treated here, up to the 4-loop level. This is meant as a concise reference for practitioners in the field.

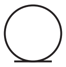
Acknowledgments

We thank M. Kalmykov for comments on the 3-loop integrals, S. Moch for correspondence on harmonic sums, and M. Steinhauser for reading the manuscript. Y.S. would like to thank J.M. Gelinas and the MIT/LNS computer services group for their efforts with installing Condor [31], which helped finding the difference equations needed for this work in finite time. A.V. was supported in part by the Foundation of Magnus Ehrnrooth.

A. Numerical results for analytically known master integrals

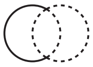
As a complement to section 4, we here list the first few terms of the Laurent expansions in $\epsilon = (4-d)/2$ of those single-mass-scale vacuum master integrals up to four loops that are known analytically (see the explicit d-dimensional expressions of section 6).

Notation and integral measure are as in section 4, which in particular determines the 1-loop tadpole to be $J = \frac{1}{\Gamma(2+\epsilon)} \int \frac{\mathrm{d}^{4-2\epsilon}p}{\pi^{2-\epsilon}} \frac{1}{p^2+1} = \frac{-1}{\epsilon(1-\epsilon^2)} = -\sum_{n=0}^{\infty} \epsilon^{2n-1}$.

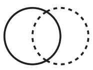


$$
\begin{aligned}
= & - 1.0000000000000000000000000000000000000000000000000 \, \epsilon^{-1} - \\
& - 1.0000000000000000000000000000000000000000000000000 \, \epsilon - \\
& - 1.0000000000000000000000000000000000000000000000000 \, \epsilon^{3} + \\
& + \mathcal{O}\left(\epsilon^{5}\right)
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
= & - 0.5000000000000000000000000000000000000000000000000 \, \epsilon^{-2} - \\
& - 0.5000000000000000000000000000000000000000000000000 \, \epsilon^{-1} - \\
& - 3.6449340668482264364724151666460251892189499012068 - \\
& - 3.4428771636886321510726770051345751984539636088663 \, \epsilon - \\
& - 17.748133915933433322311939139507906328597192596121 \, \epsilon^{2} - \\
& - 16.366439374126401323669287645924253086404587829687 \, \epsilon^{3} + \\
& + \mathcal{O}\left(\epsilon^{4}\right)
\end{aligned} \tag{A.2}
$$



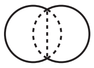
$$
\begin{aligned}
= & - 0.083333333333333333333333333333333333333333333333333 \, \epsilon^{-2} - \\
& - 0.37500000000000000000000000000000000000000000000000 \, \epsilon^{-1} - \\
& - 2.4683003667574465515695409166563459279428082839367 - \\
& - 8.5848042311088475775631523236940150167718153674315 \, \epsilon - \\
& - 38.120827450450135424466436253406610052456582985006 \, \epsilon^{2} + \\
& + \mathcal{O}\left(\epsilon^{3}\right)
\end{aligned} \tag{A.3}
$$



$$
\begin{aligned}
= & + 0.33333333333333333333333333333333333333333333333333 \, \epsilon^{-3} + \\
& + 0.16666666666666666666666666666666666666666666666667 \, \epsilon^{-2} + \\
& + 0.58333333333333333333333333333333333333333333333333 \, \epsilon^{-1} + \\
& + 0.41381840842558476106596843069719997537329677957466 - \\
& - 24.905969600320865917659060143145414845610363033237 \, \epsilon - \\
& - 12.059724940640299353325034075589267393005352211165 \, \epsilon^{2} + \\
& + \mathcal{O}\left(\epsilon^{3}\right)
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
= & - 0.33333333333333333333333333333333333333333333333333 \, \epsilon^{-3} - \\
& - 0.66666666666666666666666666666666666666666666666667 \, \epsilon^{-2} - \\
& - 5.9565348003631195396114969999587170451045664690803 \, \epsilon^{-1} - \\
& - 10.976993729846780032023343117902167435855817881707 - \\
& - 67.587197404302297868575437012376235190940093056288 \, \epsilon -
\end{aligned}
$$

$$
\begin{aligned}
&- 120.04368176952781664333978568556636327245719008030\ \epsilon^2 + \\
&+ \mathcal{O}\left(\epsilon^3\right) \qquad \text{(A.5)}
\end{aligned}
$$



$$
\begin{aligned}
= &+ 0.08333333333333333333333333333333333333333\ \epsilon^{-3} + \\
&+ 0.2361111111111111111111111111111111111111\ \epsilon^{-2} + \\
&+ 0.4189814814814814814814814814814814814815\ \epsilon^{-1} + \\
&+ 0.5870437170675600697079437393391752840153 - \\
&- 38.15649063807203021274518890590965524153\ \epsilon + \mathcal{O}\left(\epsilon^2\right) \qquad \text{(A.6)}
\end{aligned}
$$

$$
\begin{aligned}
= &+ 0.1666666666666666666666666666666666666667\ \epsilon^{-4} + \\
&+ 0.1666666666666666666666666666666666666667\ \epsilon^{-3} - \\
&- 0.1666666666666666666666666666666666666667\ \epsilon^{-2} + \\
&+ 0.1130270533440971408650083562391832748449\ \epsilon^{-1} - \\
&- 78.06045784302350379977045250776444270422 - \\
&- 77.68037792722049725359022296531379645497\ \epsilon + \mathcal{O}\left(\epsilon^2\right) \qquad \text{(A.7)}
\end{aligned}
$$

In fact, the results shown here have been obtained via numerically evaluating truncated factorial series along the lines of sections 2 and 3, but they of course coincide perfectly with the analytical results of section 6 (note the different normalization there).

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